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Equilibria with discontinuous preferences: New fixed point theorems [☆]

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ABSTRACT

We prove new equilibrium existence results for games and economies with discontinuous and non-ordered preferences. To do so, we introduce the notion of "continuous inclusion property", and prove new fixed point theorems which extend and generalize the results of Fan [17], Glicksberg [20], Browder [7], and Gale and Mas-Colell [19]. Our results also extend the previous work of Yannelis [47] and Wu and Shen [46].

1. Introduction

Since the pioneer works of Dasgupta and Maskin [14] and Reny [34] on the existence of Nash equilibria in games with discontinuous payoffs, a number of authors have extended their results in different directions, see for example, Lebrun [26], Bagh and Jofre [2], Monteiro and Page [29], Bich [4,5], Bich and Laraki [6], Carbonell-Nicolau [8], Carbonell-Nicolau and McLean [9], Carmona [10–12], Prokopovych [31–33], de Castro [16], Reny [34,35,37,38], Nessah and Tian [30], Scalzo [39,41], Tian [43], Uyanık [44], and He and Yannelis [21,23,24].[1]

In this paper, we provide new equilibrium existence results for discontinuous games which are not covered by the above literature. To this end, we introduce the notion of "continuous inclusion property", which allows us to prove two new fixed point theorems. The correspondences satisfying the continuous inclusion property

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[1] See He and Yannelis [22] and Carmona and Podczeck [13] for additional references. Interested readers can consult Reny [36] for the discussions about the state of the art on the subject.

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could be neither lower nor upper hemicontinuous, actually they may be discontinuous. The continuous inclusion property is a very weak condition in the sense that any correspondence, which has either an open graph, or open lower sections, or the local intersection property[2] or it is upper hemicontinuous, will automatically satisfy this property. Our first result is an extension of fixed point theorems of Fan [17] and Glicksberg [20], which also generalizes the Browder [7] fixed point theorem in locally convex spaces. The second result generalizes substantially the fixed point theorem of Gale and Mas-Colell [19].

With the help of the above two fixed point theorems, we prove several new results. Firstly, we show the nonemptiness of demand correspondences for non-ordered and discontinuous preferences. This result generalizes the theorem of Sonnenschein [42]. Secondly, we prove the existence of Nash equilibrium for discontinuous games with non-ordered preferences. This extends the results in Reny [34] to non-ordered preferences. Thirdly, we extend the classical Gale–Debreu–Nikaido lemma (see Debreu [15]) by allowing for discontinuous demand correspondences. Our extension generalizes the Gale–Debreu–Nikaido lemma to infinite dimensional spaces, and also extends the results of Aliprantis and Brown [1] and Yannelis [47]. To show that our generalization is non-vacuous, an example of a Walrasian equilibrium with discontinuous preferences is provided, which cannot be covered by any existence result in the literature. However, our version of the Gale–Debreu–Nikaido lemma can be applied to this example.

The rest of the paper is organized as follows. In Section 2, the "continuous inclusion property" is proposed, and then we prove a fixed-point theorem and a generalization of the fixed-point theorem of Gale and Mas-Colell [19]. The existence of Nash equilibrium in games with discontinuous preferences is obtained as a direct corollary. Section 3 collects the generalization of the Gale–Debreu–Nikaido lemma to the setting with discontinuous preferences in infinite dimensional spaces.

2. Results

2.1. Definitions

Let X and Y be linear topological spaces. Suppose that ψ is a correspondence from X to Y. Then ψ is said to be **upper hemicontinuous** if the upper inverse $\psi^u(V) = \{x \in X : \psi(x) \subseteq V\}$ is open in X for every open subset V of Y, and **upper demicontinuous** if the upper inverse of every open half space in Y is open in X. The correspondence ψ is said to be **lower hemicontinuous** if the lower inverse $\psi^l(V) = \{x \in X : \psi(x) \cap V \neq \emptyset\}$ is open in X for every open subset V of Y. In addition, if $\psi^l(y) = \{x \in X : y \in \psi(x)\}$ is open for each $y \in Y$, then ψ is said to have **open lower sections**. At some $x \in X$, if there exists an open set O_x such that $x \in O_x$ and $\cap_{x' \in O_x} \psi(x') \neq \emptyset$, then we say that ψ has the local intersection property. Furthermore, ψ is said to have the **local intersection property** if this property holds for every $x \in X$.[3] Given a linear topological space X, its dual is the space X^* of all continuous linear functionals on X.

We now introduce the following "continuous inclusion property".

Definition 1. A correspondence ψ from X to Y is said to have the **continuous inclusion property** at x if there exists an open neighborhood O_x of x and a nonempty correspondence $F_x : O_x \to 2^Y$ such that $F_x(z) \subseteq \psi(z)$ for any $z \in O_x$ and coF_x[4] has a closed graph.[5]

[2] See Wu and Shen [46].

[3] A continuous selection exists if a correspondence has open lower sections (see Yannelis and Prabhakar [48]) or the local intersection property (see Wu and Shen [46]) under certain convexity conditions. Scalzo [40] proposed the "local continuous selection property", which is necessary and sufficient for the existence of a continuous selection.

[4] For a correspondence F, co$F$ is the convex hull of F.

[5] If the sub-correspondence F_x has a closed graph and X is finite dimensional, then coF_x still has a closed graph since the convex hull of a closed set is closed in finite dimensional spaces. However, this may not be true if one works with infinite dimensional spaces. One can easily see that assuming the sub-correspondence F_x is convex valued and has a closed graph would suffice for our aim.

Notice that if a correspondence has the continuous inclusion property, then at every point of its domain the correspondence includes a sub-correspondence such that the convex hull of this sub-correspondence is locally closed.

Remark 1. It can be easily checked that any nonempty correspondence with open lower sections has the local intersection property, and any correspondence with the local intersection property has the continuous inclusion property. Furthermore, any upper hemicontinuous, convex and compact valued correspondence satisfies the continuous inclusion property.

Below, we provide an example of a correspondence which satisfies the continuous inclusion property, but does not have open lower sections or the local intersection property.

Example 1. Let

$$F(x) = \begin{cases} [0,1], & x = 0; \\ \{|\sin(\frac{1}{x})|, |\cos(\frac{1}{x})|\}, & x \in (0,1]. \end{cases}$$

The functions $|\sin(\frac{1}{x})|$ and $|\cos(\frac{1}{x})|$ oscillate in any neighborhood of the origin and will pass every point of the unit interval for infinitely many times as x approaches 0. Notice that F satisfies the continuous inclusion property as it is upper hemicontinuous.

The correspondence F does not have any continuous selection. Otherwise, suppose that F has a continuous selection f and $f(0) = a \in [0,1]$. Then for sufficiently small $\delta > 0$, we have $f(x) \in [a - 0.001, a + 0.001]$ for $x \in (0, \delta)$. Pick some $b \in (0,1)$ and z_0 such that $b, \sqrt{1-b^2} \notin [a - 0.01, a + 0.01]$ and $\sin(z_0) = b$. Let $x_k = \frac{1}{z_0 + 2k\pi}$. Then for sufficiently large k, $x_k \in (0, \delta)$. However, $f(x_k) \in \{|\sin(\frac{1}{x_k})|, |\cos(\frac{1}{x_k})|\} = \{b, \sqrt{1-b^2}\}$, which is a contradiction.

It is obvious that $F^l(y)$ is not open for any $y \in [0,1]$, and F does not have the local intersection property. In particular, for any $x \in (0,1)$ and $\delta > 0$, there exist two distinct points x' and x'' such that $|x - x'| \leq \delta$, $|x - x''| \leq \delta$, and $F(x') \cap F(x'') = \emptyset$.

2.2. Operations on correspondences

In this subsection, we consider the preservation and the failure of the continuous inclusion property under some usual operations, including union, inclusion, addition and product.

Let X, Y, Z, $\{X_j\}_{j \in J}$ and $\{Y_j\}_{j \in J}$ be linear topological spaces, where J is an index set. Given a family of correspondences $\{\psi_j\}_{j \in J}$ from X to Y, we define the union and intersection of this family pointwise. That is, $\cup_{j \in J} \psi_j$ maps x to $\cup_{j \in J} \psi_j(x)$, and $\cap_{j \in J} \psi_j$ maps x to $\cap_{j \in J} \psi_j(x)$.

Let ψ_1 and ψ_2 be two correspondences from X to Y, and α and β be two nonzero numbers. The linear combination $\alpha\psi_1 + \beta\psi_2$ of ψ_1 and ψ_2 is defined as

$$(\alpha\psi_1 + \beta\psi_2)(x) = \{\alpha y_1 + \beta y_2 \colon y_1 \in \psi_1(x), y_2 \in \psi_2(x)\}.$$

The product of a family of correspondences $\{\psi_j \colon X_j \to 2^{Y_j}\}_{j \in J}$ is the correspondence $\prod_{j \in J} \psi_j$ from $\prod_{j \in J} X_j$ to $\prod_{j \in J} Y_j$, defined naturally by $(\prod_{j \in J} \psi_j)(x) = \prod_{j \in J} \psi_j(x_j)$ for each $x = \{x_j\}_{j \in J}$.

In the next proposition, we consider the preservation and the failure of the continuous inclusion property under some regularity conditions.

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Proposition 1.

1. *Let $\psi_1\colon X \to 2^Y$ be a correspondence having the continuous inclusion property, and $\{\phi_j\colon X \to 2^Y\}_{j\in J}$ be a family of arbitrary correspondences. Then their union $(\cup_{j\in J}\phi_j)\cup\psi_1$ also has the continuous inclusion property.*
2. *Let $\psi_1\colon [0,1] \to 2^{[0,1]}$ and $\psi_2\colon [0,1] \to 2^{[0,1]}$ be two correspondences both having the continuous inclusion property, their intersection may not have the continuous inclusion property.*
3. *If Y is a compact Hausdorff space, and $\psi,\phi\colon X \to 2^Y$ are convex valued correspondences with the continuous inclusion property, then $\alpha\psi+\beta\phi$ has the continuous inclusion property for any nonzero α and β.*
4. *Let $\{\psi_i\colon X_i \to 2^{Y_i}\}_{1\le i\le n}$ be a finite family of correspondences having the continuous inclusion property. Then their product $\prod_{1\le i\le n}\psi_i$ also has the continuous inclusion property.*

Proof. (1) Fix $x \in X$. Since ψ_1 has the continuous inclusion property, there exists an open neighborhood O_x of x and a nonempty correspondence $F_x\colon O_x \to 2^Y$ such that $F_x(x') \subseteq \psi_1(x')$ for any $x' \in O_x$ and coF_x has a closed graph. Since ψ_1 is a sub-correspondence of $(\cup_{j\in J}\phi_j)\cup\psi_1$, the rest is clear.

(2) Let $\psi_1\colon [0,1] \to 2^{[0,1]}$ and $\psi_2\colon [0,1] \to 2^{[0,1]}$ be defined as follows:

$$\psi_1(x) = \begin{cases} \{x,0\}, & 0 \le x \le \frac{1}{2}, \\ \{x,1\}, & \frac{1}{2} < x \le 1; \end{cases} \qquad \psi_2(x) = \begin{cases} \{1-x,0\}, & 0 \le x \le \frac{1}{2}; \\ \{1-x,1\}, & \frac{1}{2} < x \le 1. \end{cases}$$

It is obvious that ψ_1 and ψ_2 satisfy the continuous inclusion property since both of them have continuous selections. However, their intersection is

$$\psi_1 \cap \psi_2(x) = \begin{cases} \{0\}, & 0 \le x < \frac{1}{2}; \\ \{0, \frac{1}{2}\}, & x = \frac{1}{2}; \\ \{1\}, & \frac{1}{2} < x \le 1. \end{cases}$$

It is clear that the correspondence $\psi_1 \cap \psi_2$ does not satisfy the continuous inclusion property at the point $\frac{1}{2}$.

(3) Fix $x \in X$. Since ψ and ϕ are convex valued and have the continuous inclusion property at x, there exist open neighborhoods O_x^1 and O_x^2 of x, and nonempty convex valued correspondences $F_x^1\colon O_x^1 \to 2^Y$ and $F_x^2\colon O_x^2 \to 2^Y$ such that $F_x^1(x') \subseteq \psi(x')$ for any $x' \in O_x^1$ and $F_x^2(x') \subseteq \phi(x')$ for any $x' \in O_x^2$, and both F_x^1 and F_x^2 have closed graphs. Let $O_x = O_x^1 \cap O_x^2$ and $G_x = \alpha F_x^1 + \beta F_x^2$. Then O_x is an open neighborhood of x, G_x is convex valued, and $G_x(x') \subseteq (\alpha\psi+\beta\phi)(x')$ for any $x' \in O_x$. Since Y is a compact Hausdorff space and F_x^1 (resp. F_x^2) has a closed graph, F_x^1 (resp. F_x^2) is upper hemicontinuous and compact valued. As a result, G_x is upper hemicontinuous and compact valued, and hence has a closed graph. This proves our claim.

(4) This property is obvious. □

2.3. A fixed-point theorem

Below we prove a fixed-point theorem based on the continuous inclusion property. The Fan–Glicksberg theorem shows that an upper hemicontinuous, nonempty, convex and compact valued correspondence has a fixed point under some regularity conditions. Our theorem replaces the upper hemicontinuity condition on the fixed point theorems of Fan [17] and Glicksberg [20] by the continuous inclusion property. Since an upper hemicontinuous, convex and compact valued correspondence has the continuous inclusion property, our fixed point theorem improves the fixed point theorems of Fan [17] and Glicksberg [20].

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Theorem 2. *Let X be a nonempty, compact, convex subset of a Hausdorff locally convex linear topological space Y, and $\psi\colon X \to 2^X$ be a correspondence which is nonempty and convex valued, and has the continuous inclusion property. Then there exists a point $x^* \in X$ such that $x^* \in \psi(x^*)$.*

Proof. Since ψ has the continuous inclusion property, for each $x \in X$, there exists an open neighborhood O_x and a nonempty correspondence $F_x\colon O_x \to 2^X$ such that $F_x(z) \subseteq \psi(z)$ for any $z \in O_x$ and $\text{co}F_x$ has a closed graph.

The collection $\mathscr{C} = \{O_x\colon x \in X\}$ is an open cover of X. Since X is compact, there is a finite set $\{x_1, \dots, x_n\}$ such that $X \subseteq \cup_{1\le i\le n} O_{x_i}$. Let $\{E_{x_i}\}_{1\le i\le n}$ be a closed refinement; that is, $E_{x_i} \subseteq O_{x_i}$, E_{x_i} is closed and $X = \cup_{1\le i\le n} E_{x_i}$ (see Michael [27, Lemma 1]).

For each $x \in X$, let $I(x) = \{1 \le i \le n\colon x \in E_{x_i}\}$, and $F(x) = \text{co}\left(\cup_{i\in I(x)} \text{co}F_{x_i}(x)\right)$. Then it is obvious that F is nonempty and convex valued. Moreover, F is also compact valued; see Hildenbrand [25, p. 37]. For each x and $i \in I(x)$, $F_{x_i}(x) \subseteq \psi(x)$. Since ψ is convex valued, $\text{co}F_{x_i}(x) \subseteq \psi(x)$, which implies that $\cup_{i\in I(x)} \text{co}F_{x_i}(x) \subseteq \psi(x)$. Again by the convexity of $\psi(x)$, we have $F(x) = \text{co}\left(\cup_{i\in I(x)} \text{co}F_{x_i}(x)\right) \subseteq \psi(x)$.

Since $\text{co}F_{x_i}$ has a closed graph in E_{x_i} and X is a compact Hausdorff space, it is upper hemicontinuous in E_{x_i}. We can slightly abuse the notation by assuming that $\text{co}F_{x_i}$ is empty when $x_i \notin E_{x_i}$. As E_{x_i} is a closed set, the correspondence $\text{co}F_{x_i}$ is upper hemicontinuous on the whole space. For each x, $I(x)$ is finite, which implies that $\cup_{i\in I(x)} \text{co}F_{x_i}(x)$ is the union of a finite family of upper hemicontinuous correspondences, and hence is upper hemicontinuous (see Hildenbrand [25, p. 22]). Since $F(x)$ is the convex hull of $\cup_{i\in I(x)} \text{co}F_{x_i}(x)$ and it is compact valued, it is also upper hemicontinuous (see Proposition 6 in Hildenbrand [25, p. 26]). By Fan–Glicksberg's fixed-point theorem (see Fan [17] and Glicksberg [20]), there is a point $x^* \in X$ such that $x^* \in F(x^*) \subseteq \psi(x^*)$. □

We provide a simple example to illustrate the above result.

Example 2. Let

$$F(x) = \begin{cases} \left(\frac{1}{1+2x}, \frac{1}{1+x}\right), & 0 \le x < \frac{1}{2}; \\ \left(\frac{1}{1+5x}, \frac{1}{1+4x}\right), & 1 \ge x > \frac{1}{2}; \\ [0,1], & x = \frac{1}{2}. \end{cases}$$

It is easy to see that the correspondence F does not have a continuous selection, and is not closed valued. As a result, the fixed point theorems of Brouwer and Kakutani are not readily applicable. However, F has the continuous inclusion property as it includes an upper hemicontinuous sub-correspondence.

Remark 2. Browder [7, Theorem 1] and Yannelis and Prabhakar [48, Theorem 3.3] proved a fixed point theorem by assuming that Y a Hausdorff linear topological space (not necessarily locally convex) and the correspondence ψ has open lower sections. In Wu and Shen [46, Theorem 2], Y is required to be locally convex and ψ has the local intersection property. Since the local intersection property implies the continuous inclusion property, our result covers the theorem of Wu and Shen [46] as a corollary.

The continuous inclusion property requires that the correspondence ψ has a locally closed sub-correspondence. As shown in the proof, this implies that the correspondence ψ contains a globally upper hemicontinuous sub-correspondence. Such a majorization idea has been adopted in Wu [45] to generalize the result of Michael [28].

2.4. A generalization of the Gale and Mas-Colell's fixed-point theorem

Below, we will generalize the fixed-point theorem of Gale and Mas-Colell [19] based on our continuous inclusion property.

Theorem 3. *Let I be a countable set and for each $i \in I$, X_i be a nonempty, compact, convex and metrizable subset of a Hausdorff locally convex linear topological space, and $X = \prod_{i \in I} X_i$. For each $i \in I$, let $\psi_i\colon X \to 2^{X_i}$ be a convex valued correspondence, and $I(x) = \{i \in I\colon \psi_i(x) \neq \emptyset, x_i \notin \psi_i(x)\}$. Suppose that for every $x \in X$ with $I(x) \neq \emptyset$, there is some $i \in I(x)$ such that ψ_i has the continuous inclusion property at x. Then there exists a point $x^* \in X$ such that for each i, either $x_i^* \in \psi_i(x^*)$ or $\psi_i(x^*) = \emptyset$.*

We first present two preparatory lemmas.

Lemma 1. *Suppose that the conditions in Theorem 3 hold. For each i, let*

$$U_i = \{x \in X\colon \psi_i \text{ has the continuous inclusion property at } x\}.$$

If $U_i = \emptyset$ for all i, then the result of Theorem 3 is true.

Proof. Since $U_i = \emptyset$ for each i, $I(x) = \emptyset$ for all x by the conditions in Theorem 3, which implies that for each i, either $x_i \in \psi_i(x)$ or $\psi_i(x) = \emptyset$. □

Lemma 2. *Under conditions of Theorem 3, for each i such that $U_i \neq \emptyset$, there exists a nonempty, convex and compact valued, upper hemicontinuous correspondence $\phi_i\colon U_i \to 2^{X_i}$ such that $\phi_i(x) \subseteq \psi_i(x)$ for each $x \in U_i$.*

Proof. Suppose that $U_i \neq \emptyset$. Since ψ_i has the continuous inclusion property at each $x \in U_i$, there exists an open subset $O_x^i \subseteq X$ such that $x \in O_x^i$ and a correspondence $F_x^i\colon O_x^i \to 2^{X_i}$ with nonempty values such that $F_x^i(z) \subseteq \psi_i(z)$ for any $z \in O_x^i$ and $\mathrm{co}F_x^i$ is closed. Then $O_x^i \subseteq U_i$, which implies that U_i is open. Since X is metrizable, U_i is paracompact (see for example, Michael [28, p. 831]). Moreover, the collection $\mathscr{C}_i = \{O_x^i\colon x \in X\}$ is an open cover of U_i. There is a closed locally finite refinement $\mathcal{F}_i = \{E_k^i\colon k \in K\}$, where K is an index set and E_k^i is a closed set in X (see Michael [27, Lemma 1]).

For each $k \in K$, choose $x_k \in X$ such that $E_k^i \subseteq O_{x_k}^i$. For each $x \in U_i$, let $I_i(x) = \{k \in K\colon x \in E_k^i\}$. Then $I_i(x)$ is finite for each $x \in U_i$. Let $\phi_i(x) = \mathrm{co}\left(\cup_{k \in I_i(x)} \mathrm{co}F_{x_k}^i(x)\right)$ for $x \in U_i$. For each x and $k \in I_i(x)$, $F_{x_k}^i(x) \subseteq \psi_i(x)$. Thus, $\mathrm{co}F_{x_k}^i(x) \subseteq \psi_i(x)$, which implies that $\cup_{k \in I_i(x)} \mathrm{co}F_{x_k}^i(x) \subseteq \psi_i(x)$. As a result, we have $\phi_i(x) = \mathrm{co}\left(\cup_{k \in I_i(x)} \mathrm{co}F_{x_k}^i(x)\right) \subseteq \psi_i(x)$.

Since $\mathrm{co}F_{x_k}^i$ has a closed graph in E_k^i and X_i is a compact Hausdorff space, $\mathrm{co}F_{x_k}^i$ is compact valued and upper hemicontinuous. For each x, $I_i(x)$ is finite, which implies that $\cup_{k \in I_i(x)} \mathrm{co}F_{x_k}^i(x)$ is the union of values for a finite family of compact valued and upper hemicontinuous correspondences, and hence is also compact valued and upper hemicontinuous at the point x. Since each $\mathrm{co}F_{x_k}^i(x)$ is convex and compact, and $\phi_i(x)$ is the convex hull of the finite union $\cup_{k \in I_i(x)} \mathrm{co}F_{x_k}^i(x)$, $\phi_i(x)$ is also compact, which implies that $\phi_i(x)$ is upper hemicontinuous at the point $x \in U_i$. This completes the proof. □

Now we are ready to prove Theorem 3.

Proof of Theorem 3. By Lemma 1, we only need to consider the case that there exists some i such that $U_i \neq \emptyset$.

Define a correspondence

$$H_i(x) = \begin{cases} \phi_i(x), & x \in U_i; \\ X_i, & \text{otherwise}. \end{cases}$$

Then it is obvious that H_i is nonempty, convex and compact valued. Since U_i is open and ϕ_i is upper hemicontinuous by Lemma 2, H_i is upper hemicontinuous on the whole space. Let $H = \prod_{i \in I} H_i$. Since H

is nonempty, convex and compact valued, and upper hemicontinuous, by the Fan–Glicksberg fixed point theorem (see Fan [17] and Glicksberg [20]), there exists a point $x^* \in X$ such that $x^* \in H(x^*)$.

Let $J = \{i \in I : x_i^* \notin \psi_i(x^*)\}$. Then $I(x^*) \subseteq J$. If $x^* \in U_j$ for some $j \in J$, then $x_j^* \in \phi_j(x^*) \subseteq \psi_j(x^*)$, which is a contradiction. Thus, we have $x^* \notin U_j$ for every $j \in J$, which implies that $I(x^*) = \emptyset$. Therefore, for every $j \in J$, $\psi_j(x^*) = \emptyset$. For every $i \in I \setminus J$, $x_i^* \in \psi_i(x^*)$. The proof is complete. □

Remark 3. In Gale and Mas-Colell [19], X_i is finite dimensional and ψ_i is lower hemicontinuous for each i. Then the continuous selection theorem of Michael [28, Theorem 3.1”’] implies that ψ_i has a continuous selection ϕ_i on U_i, which can be regarded as a continuous sub-correspondence of ψ_i. Thus, the continuous inclusion property holds and the result follows.

In addition, our result implies that one can further weaken the lower hemicontinuity condition of Gale and Mas-Colell [19]. Specifically, at each $x \in U_i$, suppose that there exists an open neighborhood O_x^i of x and a nonempty convex valued, lower hemicontinuous correspondence $F_x^i : O_x^i \to X_i$ with $F_x^i(z) \subseteq \psi_i(z)$ for $z \in O_x^i$. Then the continuous inclusion property still holds. However, in this case X_i is still required to be finite dimensional since the continuous selection theorem of Michael [28] is needed.

2.5. Existence of maximal elements

Suppose that X is a nonempty subset of a linear topological space. Let $P(x) = \{y \in X : (y,x) \in \mathscr{P}\}$ for all $x \in X$, where $\mathscr{P}$ is some binary relation on X. Then P is a preference correspondence induced by $\mathscr{P}$ on X. If $P(x^*) = \emptyset$ for some $x^* \in X$, then x^* is said to be a **maximal element** in X.

Corollary 1. *Let X be a compact, convex subset of a Hausdorff locally convex linear topological space and $P : X \to 2^X$ be a correspondence such that for all $x \in X$, $x \notin coP(x)$. If P has the continuous inclusion property at each $x \in X$ such that $P(x) \neq \emptyset$, then there exists a point $x^* \in X$ such that $P(x^*) = \emptyset$.*

Proof. By way of contradiction, suppose that $P(x) \neq \emptyset$ for all $x \in X$. Then the correspondence $\psi(x) = \mathrm{co}P(x)$ is convex and nonempty valued. It is clear that ψ has the continuous inclusion property. By Theorem 2, there exists a fixed point $x^* \in X$ such that $x^* \in \psi(x^*) = \mathrm{co}P(x^*)$, a contradiction. □

Remark 4. Theorem 5.1 of Yannelis and Prabhakar [48] proved the existence of maximal element when X is a compact, convex subset of a Hausdorff linear topological space and the correspondence P has open lower sections. This result generalizes Lemma 4 of Fan [18]. In our Corollary 1, the condition on the correspondence is more general while X is required to be locally convex.

Below, we shall illustrate the usefulness of the above corollary.

Let $\triangle$ and X be two Hausdorff locally convex linear topological spaces, where $\triangle$ is the set of all price vectors and X is the set of goods. Let the correspondence $B : \triangle \to 2^X$ denote the *budget set* which is assumed to be nonempty, convex and compact valued. The *preference correspondence* is denoted by $P : X \to 2^X$ and satisfies the condition that $x \notin \mathrm{co}P(x)$ for any $x \in X$. Let $\psi(p,x) = B(p) \cap P(x)$, and define the *demand correspondence* $D : \triangle \to 2^X$ by $D(p) = \{x \in B(p) : \psi(p,x) = \emptyset\}$.

Corollary 2. *If $\psi(p,\cdot)$ has the continuous inclusion property for each $p \in \triangle$,*[6] *then the demand correspondence D is nonempty valued.*

[6] The continuity inclusion property captures the case that the preference could be discontinuous. For example, people's preference on food could dramatically change if the amount goes to the zero: people will be sick or even die.

Proof. Fix $p_0 \in \triangle$. Since $x \notin \text{co}P(x)$ for any $x \in X$, it follows that $x \notin \text{co}\psi(p_0, x)$ for any $x \in B(p_0)$. Since $\psi(p_0, \cdot)\colon B(p_0) \to 2^{B(p_0)}$ has the continuous inclusion property, $B(p_0)$ is nonempty, convex and compact, by Corollary 1, there exists a point $x_0 \in B(p_0)$ such that $\psi(p_0, x_0) = \emptyset$. That is, $x_0 \in D(p_0)$, which implies that D is nonempty valued. □

The above corollary generalizes the corresponding theorem in Sonnenschein [42] by relaxing the continuity assumption.

2.6. The existence of Nash equilibrium

Below, we obtain the existence of a Nash equilibrium in games with (possibly) nontransitive, incomplete, discontinuous preferences as a direct corollary of Theorem 3. Notice that the preference need not be representable by a utility function.

Let I be a set of countable players, and the game is $\Gamma = \{(X_i, P_i)\colon i \in I\}$, where X_i is the **action space** of player i, $X = \prod_{i \in I} X_i$, and the **preference correspondence** of player i is $P_i\colon X \to 2^{X_i}$. If the preference P_i can be represented by a utility function $u_i\colon X \to \mathbb{R}$, then

$$P_i(x) = \{y_i \in X_i\colon u_i(y_i, x_{-i}) > u_i(x)\}.$$

Corollary 3. *Let $\Gamma = \{(X_i, P_i)\colon i \in I\}$ be a game such that for each $i \in I$:*

- **i** *X_i is a nonempty, compact, convex, metrizable subset of a Hausdorff locally convex linear topological space;*
- **ii** *Let $I(x) = \{i \in I\colon P_i(x) \neq \emptyset\}$. Suppose that for every $x \in X$ with $I(x) \neq \emptyset$, there exists an agent $i \in I(x)$ such that P_i has the continuous inclusion property at x and $x_i \notin coP_i(x)$.*

Then Γ has a Nash equilibrium; that is, there exists some $x^* \in X$ such that for any $i \in I$, $P_i(x^*) = \emptyset$.

Proof. Denote $\psi_i = \text{co}P_i$ for each $i \in I$. Let $I'(x) = \{i \in I\colon \psi_i(x) \neq \emptyset, x_i \notin \psi_i(x)\}$. Then for every $x \in X$ with $I'(x) \neq \emptyset$, $I(x) \neq \emptyset$. By condition (ii), there exists an agent $i \in I'(x)$ such that ψ_i has the continuous inclusion property at x.

By Theorem 3, there exists a point $x^* \in X$ such that for each i, either $x_i^* \in \psi_i(x^*)$ or $\psi_i(x^*) = \emptyset$. Then $I(x^*) = \{i \in I\colon x_i^* \in \psi_i(x^*)\}$. If $I(x^*) \neq \emptyset$, then by condition (ii), there is an agent $i \in I(x^*)$ such that P_i has the continuous inclusion property at x^* and $x_i^* \notin \psi_i(x^*)$, which is a contradiction. As a result, $I(x^*) = \emptyset$. That is, $\psi_i(x^*) = \emptyset$ for each $i \in I$, which implies that x^* is a Nash equilibrium in the game Γ. □

The following result is an immediate corollary of Corollary 3. The continuous inclusion property is directly assumed for each player.

Corollary 4. *Let $\Gamma = \{(X_i, P_i)\colon i \in I\}$ be a game such that for each $i \in I$:*

- **i** *X_i is a nonempty, compact, convex, metrizable subset of a Hausdorff locally convex linear topological space;*
- **ii** *P_i has the continuous inclusion property at each $x \in X = \times_{i \in I} X_i$ with $P_i(x) \neq \emptyset$;*
- **iii** *$x_i \notin coP_i(x)$ for all $x \in X$.*

Then Γ has a Nash equilibrium; that is, $\exists x^* \in X$ such that for any $i \in I$, $P_i(x^*) = \emptyset$.

Remark 5. Suppose that for each $i \in I$, the utility function u_i satisfies the generalized payoff security condition of Carmona [11], and define the value function $g_i\colon X_{-i} \to \mathbb{R}$ by $g_i(x_{-i}) = \sup_{x_i \in X_i} u_i(x_i, x_{-i})$. Fix $\epsilon > 0$. For each $i \in I$, consider the correspondence

$$M_i^\epsilon(x_{-i}) = \{x_i \in X_i \colon u_i(x_i, x_{-i}) > g_i(x_{-i}) - \epsilon\}.$$

Then it is easy to see that M_i^ϵ has the continuous inclusion property. Following the argument in Prokopovych [31], one can impose standard conditions (e.g., quasiconcavity and transfer reciprocal upper semicontinuity) to prove the existence of approximate and exact Nash equilibrium.

Remark 6. Reny [34] proved the existence of a pure strategy Nash equilibrium in games with discontinuous payoffs based on some payoff security type condition. Our Corollaries 3 and 4 extend his results to non-ordered preferences, but do not imply his and vice versa. However, to verify the conditions of theorems in the above paper, one has to work with the non-equilibrium point, and check for all players at every point in a neighborhood of this non-equilibrium point. To the contrary, we can check the preference correspondence for each agent separately, as shown in Corollary 4.[7]

3. A generalization of the Gale–Debreu–Nikaido lemma

In this section, using the fixed point theorem (Theorem 2), we provide a generalization of the Gale–Debreu–Nikaido lemma to an infinite-dimensional commodity space with discontinuous excess demand correspondences.

Let X be a Hausdorff locally convex linear topological space, and E a closed, convex cone of X having an interior point e. Denote $E^* = \{p \in X^* \colon p \cdot x \le 0 \text{ for all } x \in E\} \neq \{0\}$; that is, E^* is the dual cone of E. Let $\triangle = \{p \in E^* \colon p \cdot e = -1\}$ and $Z\colon \triangle \to 2^X$ be an excess demand correspondence. Given $p \in \triangle$, let $Y(p) = \{x \in X \colon p \cdot x \le 0\}$ and $\Gamma(p) = Y(p) \cap Z(p)$.

Theorem 4. *If Γ is nonempty and convex valued, and satisfies the continuous inclusion property, where X^* is endowed with the weak* topology, then $\exists p^* \in \triangle$ such that $Z(p^*) \cap E \neq \emptyset$.*

Proof. Define a correspondence Π from E to $\triangle$ as follows: for each $x \in E$,

$$\Pi(x) = \operatorname{argmax}_{p \in \triangle}(p \cdot x).$$

By Alaoglu's Theorem, $\triangle$ is weak* compact. By Berge's maximum theorem (see Berge [3]), Π is nonempty, convex and weak* compact valued, and upper hemicontinuous.

Define the correspondence Ψ from $E \times \triangle$ to $E \times \triangle$ as $\Psi(x, p) = \Gamma(p) \times \Pi(x)$ for each $(x, p) \in E \times \triangle$. It is obvious that Ψ is nonempty and convex valued. For each $p_0 \in \triangle$, since Γ is convex valued and has the continuous inclusion property, there exists a weak* open neighborhood O_{p_0} of p_0, and a nonempty, convex valued and weak* upper hemicontinuous correspondence $F_{p_0}\colon O_{p_0} \to 2^E$ such that $F_{p_0}(q) \subseteq \Gamma(q)$ for any $q \in O_{p_0}$. Let $\Phi(x, p) = F_{p_0}(p) \times \Pi(x)$ for $(x, p) \in E \times O_{p_0}$. Then Φ is a sub-correspondence of Ψ on $E \times O_{p_0}$, which is nonempty, convex-valued and upper hemicontinuous. Therefore, Ψ has the continuous inclusion property.

By Theorem 2, there exists $(x^*, p^*) \in E \times \triangle$ such that $(x^*, p^*) \in \Psi(x^*, p^*)$. That is,

1. $p^* \cdot x^* \ge p \cdot x^*$ for any $p \in \triangle$;
2. $x^* \in Z(p^*)$ and $p^* \cdot x^* \le 0$.

[7] For further results, see Reny [37] and Carmona and Podczeck [13]. See also Section 3.2 of He and Yannelis [22] for a discussion on the relationship of these papers.

Combining (1) and (2), we have $p \cdot x^* \leq p^* \cdot x^* \leq 0$ for any $p \in \triangle$, which implies that $x^* \in E$. Therefore, $Z(p^*) \cap E \neq \emptyset$ for some $p^* \in \triangle$. □

Below, we provide an alternative proof using Corollary 1.

Alternative Proof. Since Γ has the continuous inclusion property, for each $p \in \triangle$, there exists an open neighborhood O_p and a nonempty correspondence $G_p\colon O_p \to 2^X$ such that $G_p(q) \subseteq \Gamma(q)$ for any $q \in O_p$ and $\mathrm{co}G_p$ has a closed graph. As in the proof of Theorem 2, one can find a nonempty, convex and compact valued, weak* upper hemicontinuous correspondence $G\colon \triangle \to 2^X$ which is a sub-correspondence of Γ. Define the correspondence $F\colon \triangle \to 2^{\triangle}$ by $F(p) = \{q \in \triangle\colon q \cdot x > 0 \text{ for all } x \in G(p)\}$. Fix $q \in \triangle$. As in the proof of Yannelis [47, Theorem 3.1], one can easily show that $W = F^l(q)$, where $W = \{p \in \triangle\colon G(p) \subseteq V_q\}$ and $V_q = \{x \in X\colon q \cdot x > 0\}$. The set W is weak* open since G is weak* upper hemicontinuous. Consequently, F has weak* open lower sections, and hence has the continuous inclusion property (recall Remark 1).[8] In addition, by the definition of F, $p \notin F(p)$ for every $p \in \triangle$. Since $\triangle$ is nonempty, convex and weak* compact, by Corollary 1, there exists a point $p^* \in \triangle$ such that $F(p^*) = \emptyset$; that is,

$$\text{for any } q \in \triangle, \exists x \in G(p^*), q \cdot x \leq 0. \tag{1}$$

We will show that (1) implies $Z(p^*) \cap E \neq \emptyset$ for some $p^* \in \triangle$. Suppose otherwise, then there exists a continuous linear functional which strictly separates the convex compact set $G(p^*) \subseteq Z(p^*)$ from the closed convex set E; that is,

$$\text{there exists } r \in X^*, r \neq 0 \text{ such that } \inf_{x \in G(p^*)} r \cdot x > \sup_{x \in E} r \cdot x \geq 0. \tag{2}$$

Without loss of generality, we can take r to be in $\triangle$.[9] It follows from (2) that $r \cdot x > 0$ for any $x \in G(p^*)$, a contradiction to (1).

Therefore, $Z(p^*) \cap E \neq \emptyset$ for some $p^* \in \triangle$. □

3.1. An example

Below, we provide an example which indicates how Theorem 4 can be used to prove the existence of an equilibrium. Notice that the preferences of both agents below are neither upper hemicontinuous nor lower hemicontinuous. An equilibrium exists by virtue of our Theorem 4.

Example 3. Consider the following 2-agent 2-good economy:

1. The set of available allocations for both agents are $X_1 = X_2 = [0,1] \times [0,1]$, $X = X_1 \times X_2$.
2. The initial endowments are given by $e_1 = (\frac{2}{3}, \frac{1}{3})$ and $e_2 = (\frac{1}{3}, \frac{2}{3})$.
3. For agent 1 and an allocation $x_1 = (x_1^1, x_1^2)$ and $x_2 = (x_2^1, x_2^2)$, agent 1's preference only depends on his own allocation:
   a if $x_1^1 > x_1^2$, then $P_1(x_1, x_2) = \{(y,z)\colon z > y \geq 0, y + z \geq 1\}$;
   b if $x_1^1 < x_1^2$, then $P_1(x_1, x_2) = \{(y,z)\colon y > z \geq 0, y + z \geq 1\}$;
   c if $x_1^1 = x_1^2$, then $P_1(x_1, x_2) = \{(y,y)\colon y > x_1^1\}$.
   The preference of agent 2 is defined similarly.

Note that P_i is neither upper hemicontinuous nor lower hemicontinuous, $i = 1, 2$.

[8] The continuous inclusion property of F holds on the subset $\triangle$ of X^*, which is endowed with the weak* topology.
[9] If $r \notin \triangle$, then the fact e is an interior point of E implies that $r \cdot e < 0$, and we can replace r by $\frac{r}{-r \cdot e}$.

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For any price $p = (p_1, 1 - p_1)$, the budget set of agent 1 is

$$B_1(p) = \{(x_1^1, x_1^2) \in X_1 \colon p_1 \cdot x_1^1 + (1 - p_1) \cdot x_1^2 \leq \frac{1}{3}(1 + p_1)\},$$

and the budget set of agent 2 is

$$B_2(p) = \{(x_2^1, x_2^2) \in X_2 \colon p_1 \cdot x_2^1 + (1 - p_1) \cdot x_2^2 \leq \frac{1}{3}(2 - p_1)\}.$$

The demand correspondence for agent i is defined as

$$D_i(p) = \{x \in B_i(p) \colon P_i(x) \cap B_i(p) = \emptyset\}.$$

It is easy to see that D_i is nonempty and convex valued. In addition, given any price p, $x_1 = (\frac{1+p_1}{3}, \frac{1+p_1}{3}) \in D_1(p)$ and $x_2 = (\frac{2-p_1}{3}, \frac{2-p_1}{3}) \in D_2(p)$.

One can define functions ψ_1 and ψ_2 such that $\psi_1(p) = (\frac{1+p_1}{3}, \frac{1+p_1}{3})$ and $\psi_2(p) = (\frac{2-p_1}{3}, \frac{2-p_1}{3})$. Since $\psi_i(p) \in D_i(p)$ for every p, D_i has the continuous inclusion property for any i. Then $D_1 + D_2$ also satisfies the continuous inclusion property, Theorem 4 can be used to show the existence of an equilibrium. Indeed, $(x_1^1, x_1^2) = (x_2^1, x_2^2) = (\frac{1}{2}, \frac{1}{2})$ and $(p_1, p_2) = (\frac{1}{2}, \frac{1}{2})$ is an equilibrium.

3.2. Remarks

We show that Theorem 4 implies the standard Gale–Debreu–Nikaido lemma, see Debreu [15].

Corollary 5. *Let $X = \mathbb{R}^l$ and $Z \colon \triangle \to 2^X$ be an excess demand correspondence satisfying the following conditions:*

1. *Z is nonempty, convex and compact valued, and upper hemicontinuous;*
2. *for every $p \in \triangle$, $\exists z \in Z(p)$ such that $p \cdot z \leq 0$.*

Then, $\exists p^* \in \triangle$ such that $Z(p^*) \cap \mathbb{R}^l_- \neq \emptyset$.

Proof. Given $p \in \triangle$, let $Y(p) = \{z \in \mathbb{R}^l \colon p \cdot z \leq 0\}$ and $X(p) = Y(p) \cap Z(p)$. Due to (2), X is nonempty. Since both Y and Z are convex valued and upper hemicontinuous, X is also convex valued and upper hemicontinuous. Thus, X has the continuous inclusion property. Then the result follows from Theorem 4. □

Remark 7. Yannelis [47] proved the market equilibrium theorem of Gale–Debreu–Nikaido for an infinite dimensional commodity space by assuming that the excess demand correspondence is upper demicontinuous. In our theorem, the excess demand correspondence may not be continuous, hence not upper demicontinuous.

Suppose that X is an AM-space with the unit e, X_+ is the positive cone of X, and $\triangle = \{p \in X_+^* \colon p \cdot e = 1\}$. Aliprantis and Brown [1] worked with an excess demand function $Z \colon \triangle \to X$ instead of an excess demand correspondence, and proved the following result.

Corollary 6 *(Aliprantis and Brown [1]). Suppose that*

1. *there exists a consistent locally convex topology on X such that Z is weak* continuous;*
2. *Z satisfies the Walras law, i.e., $p \cdot Z(p) = 0$ for all $p \in \triangle$.*

Then there exists a point $p \in \triangle$ such that $Z(p) \leq 0$.

It is obvious that this result is covered by our Theorem 4, since $\Gamma(p) = Z(p)$ in their setting. As a consequence, Γ is a weak* continuous function and the continuous inclusion property automatically holds.

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